07

407

Jo2h

THE JOURNAL

OF THE

INDIAN MATHEMATICAL SOCIETY

21 APR 1933

Editor :

R. VAIDYANATHASWAMY, M.A., D.Sc.

Joint Editor :

A. NARASINGA RAO, M.A., L.T.

Collaborators :

K. ANANDA RAO, M.A.; H. RAFAEL, D.Sc., S.J.;
S. C. DHAR, D.Sc.; S. R. U. SAVOOR, M.A., D.Sc.;
V. RAMASWAMI AIYAR, M.A.; T. SURYANARAYANA, M.A.

Volume XIX. No 11.

October 1932.

MADRAS :

PRINTED BY SRINIVASA VARADACHARI & CO.
190, MOUNT ROAD.

Annual Subscription : Rs. 9.] [Single Copy : Rupee One and Annas Eight.

CONTENTS.

	PAGES
Invariant-Factors of a certain class of Linear Substitutions: S. Krishnamurthi Rao, B.A. (Hons.) ...	233—240
An Analogue of a Theorem of Phragmen-Lindelof: S. P. Jain, M.Sc.	241—245
Cyclic Curves on an Ellipsoid: S. Mukhopadhyaya ...	246—250
Remarks on Spurious Correlation: C. N. Srinivasiengar.	251—252
On $(2n-2)!/n!(n+r)!$: P. Jagannathan ...	253—256
Distribution of Height, Weight and Chest-girth in Madras Males: N. K. Adyanthaya, B.Sc. (Lond.) and N. Sundararama Sastri, M.A. ...	257—264
NOTES AND QUESTIONS:	
A Theorem on Centroids: B. B. Bagi ...	*249—250*
Method of calculating the 'final yield,' annual or half-yearly, of bonds redeemable at par after a certain number of years: M. N. Kuka ...	*251—255*
Lecture Notes—(1) The conic in areal co-ordinates: Hukam Chand	*256—258*
(2) On the Determination of the Real Foci of a Real Conic: B. B Bagi.	*259—261*
Solutions	*262—263*
Questions for Solution	*264*

A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

The Journal is open to contributions from members as well as subscribers. The Editors may also accept contributions from others.

Contributors to Part I will be presented 25 copies of reprints of their papers. Extra copies will be supplied, if desired, at net cost.

All contributions should be written legibly on one side only of the paper, and all diagrams should be neatly and accurately drawn on separate slips.

All communications intended for the Journal should be addressed to the Hon. Joint Secretary and Editor, R. VAIDYANATHASWAMY, M.A., D.Sc., Reader in Mathematics, University of Madras, Madras, or to the Joint Editor, A. NARASINGA RAO, M.A., Reader in Mathematics, Annamalai University, Annamalainagar.

All business communications regarding the Journal should be addressed to the Hony. Assistant Secretary, S. MAHADEVA AIYAR, M.A., Presidency College, Madras.

Enquiries by intending members of the Society and all other communications should be addressed to the Hony. Joint Secretary, S. B. BELEKAR, M.A., Professor of Mathematics, College of Science, Nagpur.

Also solved by M. V. Seshadri, S. V. Minakshisundaram, Ram Gopal and B. Ramamurthi.

[This question has been solved in the course of the Author's paper on 'some theorems relating to a net of plane cubic curves having three given asymptotes.' *Vide* pages 60 et seq, Part I, Vol. XVIII of this Journal. Ed.]

Question 1522.

(A. NARASINGA RAO):—Show that the ortho-centre of any triangle circumscribed to a tri-cusp hypocycloid is the circum-centre of its residual triangle (the triangle formed by the remaining tangents from the vertices).

Solution by M. V. Seshadri and B. Ramamurthi.

Let ABC be a triangle circumscribing the hypocycloid, O its ortho-centre and AP, BQ and CR the remaining tangents from A, B and C forming a triangle LMN. It is required to show that O is the circum-centre of LMN.

It has been proved in the solution of Q. 1376 on page 41, Part II, Vol. XVIII, of this journal that AP, BQ, and CR make equal angles with BC, CA and AB respectively. Let this angle be α.

Then if AP and BQ meet at L, we have

$$O\hat{A}L = O\hat{B}L.$$

$\therefore$ OABL is cyclic,

i.e.,
$$\frac{OL}{\sin OAL} = \frac{AB}{\sin AOB} = 2R$$

(where R is the circum-radius of ABC).

$$\therefore \quad OL = 2R \sin O\hat{A}L = 2R \sin (90-\alpha) = 2R \cos \alpha.$$

Similarly $OM = 2R \cos \alpha$, as also ON.

$\therefore$ $OL = OM = ON$. Or O is the circum-centre of LMN.

Questions for Solution.

Proposers of Questions are requested to send their own solutions along with their questions.

1600. (R. VAIDYANATHASWAMY):—A system of conics pass through four points; their common self-polar triangle is ABC, and the locus of their centres is the conic S circumscribed to ABC. Prove that the envelope of their asymptotes is a tricusp which touches the line at infinity at the points at infinity on S, and touches S at three other points $A'B'C'$.

Prove that

(1) The tangents at the vertices of $A'B'C'$ are parallel to the opposite sides.

(2) The diametrically opposite points $A''B''C'$ of $A'B'C$ are the points at which conics of the four-point system touch S.

(3) A triangle DEF which is inscribed in S, and is a self-polar triangle of some conic of the four-point system is necessarily apolar to $A''B''C''$.

(4) If the conic S is projected orthogonally into a circle, $A''B''C''$ will project into the equilateral triangle pedo-parallel to ABC.

Shew also that when one of the conics of the four-point system is a circle, the envelope of the asymptotes has one of its cusps situated at the centre of the circle.

1601. (A. A. KRISHNASWAMI AYYANGAR):—$P_1, P_2, P_3, P_4, P_5, P_6$, are any six points in a plane. If C_j be the oriented circle which passes through the triad, $P_iP_jP_k$ where i, j, k are any three successive numbers (1, 2, ... 6) taken in cyclic order, and θ_{ij} the angle which the circle C_i makes with the circle C_j, prove that

$$\theta_{24} + \theta_{46} + \theta_{62} = \theta_{13} + \theta_{35} + \theta_{51} \pmod{2\pi}.$$

Generalise this theorem by projection and to n-point in a plane.

N.B.—The above theorem is a generalisation of the well-known point O-theorem.

129090

OFFICE OF THE ... OF BOOKS
21 APR 1933

INVARIANT-FACTORS OF A CERTAIN CLASS OF LINEAR SUBSTITUTIONS.*

By S. Krishnamurthy Rao, B.A. (Hons.)

The usual canonical form of a linear substitution, with a single invariant-factor, is only a particular case of the substitution

$$S \equiv (a_1 x_1 + a_2 x_2 + \ldots + a_n x_n ;\ a_1 x_2 + \ldots + a_{n-1} x_n ;\ \ldots ;\ a_1 x_n)$$

the invariant-factors of which are here investigated. It is known that if $a_2 \neq 0$, S has only one invariant-factor. But the number and nature of the invariant-factors of S do not appear to have been studied for the general case

$$a_2 = 0 = a_3 = \ldots = a_r, \qquad a_{r+1} \neq 0 \qquad \ldots \quad (1)$$

Here we propose to establish that, if condition (1) holds, and if $n = sr + t \quad (t < r)$,

(1) the number of invariant-factors is r;

(2) the indices of the invariant-factors form the following sequence, called the 'Invariant-sequence':—

$$s + 1,\ s + 1,\ \ldots \text{ up to } t \text{ terms},\ s,\ s,\ \ldots \text{ up to } (r - t) \text{ terms}.$$

I am much indebted to Dr. R. Vaidyanathaswamy for suggesting this problem to me and kindly discussing it with me.

1. Maximum number of invariant-factors of S.

When condition (1) holds, the characteristic-matrix can be written

---

* Since this paper was written, I find the same problem treated by D. E. Rutherford in his paper on 'The canonical form of a rational integral function of a matrix' published in the *Proc. Edin Math. Soc.*, Series 2, Vol. 3, Part II, July '32.

JL
Bm 2, No 7
129090
600 009 - 800 009

as

$$A \equiv \begin{vmatrix} a_1-\lambda, & 0, & \dots & 0, & a_{r+1}, & a_{r+2}, & \dots & a_n \\ 0, & a_1-\lambda, & 0, & \dots & 0, & a_{r+1}, & \dots & a_{n-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \dots & \dots & & (a_1-\lambda) \end{vmatrix}$$

If, in the rth minor of A, obtained by cancelling the first r-columns and the last r-rows, we put $\lambda = a_1$, it takes the value a_{r+1}^{n-r} and therefore does not vanish. Hence S cannot have more than r invariant-factors. Presently it will be seen that the number of invariant-factors is actually equal to this upper limit.

2. Now we shall, first of all, obtain the invariant-factors of the substitution S′ given by

$$(a_1 x_1 + x_{r+1};\ \ a_1 x_2 + x_{r+2};\ \ \dots;\ \ a_1 x_{n-r} + x_n;$$
$$a_1 x_{n-r+1};\ a_1 x_{n-r+2};\ \dots;\ a_1 x_n),$$

by reducing it to its canonical form. The scheme of transformation which effects such a reduction, consists of r sets of equations, the first t of them containing $(s+1)$ equations each, and the rest of them containing s equations each. The kth of the first t sets groups together $x_k, x_{r+k}, \dots x_{sr+k}$, and the lth of the remaining $(r-t)$ sets combines together $x_l, x_{r+l}, \dots, x_{(s-1)r+l}$. That is, the equations of transformation can be written as follows:

$$\xi_{(k-1)s+k} = x_k;\ \xi_{(k-1)s+k+1} = x_{r+k};\ \dots;\ \xi_{ks+k} = x_{sr+k}$$
$$(k = 1, 2, 3, \dots t)$$

$$\xi_{(l-1)s+t+1} = x_l;\ \xi_{(l-1)s+t+2} = x_{r+l};\ \dots;\ \xi_{ls+t} = x_{(s-1)r+l}$$
$$(l = t+1, t+2, \dots, r).$$

Hence it is clear that the transformed substitution is in the canonical form consisting of r components, the first t of them being of order

$(s+1)$ each, and the rest of order s each. Therefore the invariant-sequence is

$$s+1,\ s+1,\ \ldots \text{ up to } t \text{ terms, } s,\ s,\ \ldots \text{ up to } (r-t) \text{ terms.}$$

3. In order to complete our proof, we have to show that our substitution S can be reduced to the form S′ (§ 2). This reduction is carried out by the following scheme* of transformation, which, as before, consists of r sets of equations, the first t of them containing $(s+1)$ equations each, and the rest containing s equations each.

The equations of transformation can be written as follows:

$$\left\{\begin{array}{l} \xi_k = x_k; \\ \xi_{r+k} = a_{r+1}\, x_{r+k} + a_{r+2}\, x_{r+k+1} + \qquad \ldots \\ \xi_{2r+k} = b_2\, x_{2r+k} + c_2\, x_{2r+k+1} + \ldots \\ \quad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \\ \xi_{sr+k} = b_s\, x_{sr+k} + c_s\, x_{sr+k+1} + \ldots \\ \qquad\qquad\qquad\qquad (k = 1, 2, 3, \ldots t). \end{array}\right.$$

$$\left\{\begin{array}{l} \xi \ = x_l; \\ \xi_{r+l} = a_{r+1}\, x_{r+l} + a_{r+2}\, x_{r+l+1} + \ldots \\ \xi_{2r+l} = b_2\, x_{2r+l} + c_2\, x_{2r+l+1} + \ldots \\ \quad \ldots \qquad \ldots \qquad \ldots \\ \xi_{(s-1)\, r+l} = b_{s-1}\, x_{\overline{s-1}\, r+l} + c_{s-1}\, x_{\overline{s-1}\, r+l+1} + \ldots \\ \qquad\qquad\qquad\qquad (l = t+1, t+2, \ldots r) \end{array}\right.$$

where the b's, c's, etc., are given by equations such as

$$b_s + c_s x + d_s x^2 + \ldots \equiv (a_{r+1} + a_{r+2}\, x + a_{r+3} x^2 + \ldots)^s$$

This formula for the co-efficients can be easily established by the method of induction. Assume that this law holds up to $s = s-1$.

---

* *cf.* Hilton's '*Linear Substitutions*, Ch. I, § 9, Ex. 6.

Then, since

$$\xi_{(s-1)r+k} = b_{s-1}\, x_{(s-1)r+k} + c_{s-1}\, x_{(s-1)r+k+1} + \dots,$$

$$\xi'_{(s-1)r+k} = b_{s-1}\, x'_{(s-1)r+k} + c_{s-1}\, x'_{(s-1)r+k+1} + \dots$$

$$= b_{s-1}\,[a_1\, x_{(s-1)r+k} + a_{r+1}\, x_{sr+k} + a_{r+2}\, x_{sr+k+1} + \dots]$$

$$+ c_{s-1}\,[a_1\, x_{(s-1)r+k+1} + a_{r+1}\, x_{sr+k+1} + a_{r+2}\, x_{sr+k+2} + \dots]$$

$$+ d_{s-1}\,[a_1\, x_{(s-1)r+k+2} + a_{r+1}\, x_{sr+k+2} + a_{r+2}\, x_{sr+k+3} + \dots] + \dots$$

$$= a_1\,[b_{s-1}\, x_{(s-1)r+k} + c_{s-1}\, x_{(s-1)r+k+1} + \dots]$$

$$+ a_{r+1}\, b_{s-1}\, x_{sr+k} + (a_{r+2}\, b_{s-1} + a_{r+1}\, c_{s-1})\, x_{sr+k+1}$$

$$+ (a_{r+3}\, b_{s-1} + a_{r+2}\, c_{s-1} + a_{r+1}\, d_{s-1})\, x_{sr+k+} + \dots$$

$$= a_1\, \xi_{(s-1)r+k} + a_{r+1}\, b_{s-1}\, x_{sr+k}$$

$$+ (a_{r+2}\, b_{s-1} + a_{r+1}\, c_{s-1})\, x_{sr+k+1}$$

$$+ (a_{r+3}\, b_{s-1} + a_{r+2}\, c_{s-1} + a_{r+1}\, d_{s-1})\, x_{sr+k+2} + \dots$$

Put

$$\xi_{sr+k} = a_{r+1}\, b_{s-1}\, x_{sr+k} + (a_{r+2}\, b_{s-1} + a_{r+1}\, c_{s-1})\, x_{sr+k+1}$$

$$+ (a_{r+3}\, b_{s-1} + a_{r+2}\, c_{s-1} + a_{r+1}\, d_{s-1})\, x_{sr+k+2}$$

$$+ \dots\dots\dots$$

$$= b_s\, x_{sr+k} + c_s\, x_{sr+k+1} + d_s\, x_{sr+k+2} + \dots$$

(by assumption).

Now it is clear that b_s, c_s, d_s, etc., are the co-efficients of x^0, x^1, x^2,... in the product

$$(b_{s-1} + c_{s-1}\, x + d_{s-1}\, x^2 + \dots)\,(a_{r+1} + a_{r+2}\, x + a_{r+3}\, x^2 + \dots)$$

$$= (a_{r+1} + a_{r+2}\, x + a_{r+3}\, x^2 + \dots)^{s-1}(a_{r+1} + a_{r+2}x + a_{r+3}\, x^2 + \dots)$$

$$= (a_{r+1} + a_{r+2}\, x + a_{r+3}\, x^2 + \dots)^s.$$

Hence

$$b_s + c_s x + d_s x^2 + \dots \equiv (a_{r+1} + a_{r+2} x + a_{r+3} x^2 + \dots)^s.$$

Thus we have proved that the formula for the co-efficients holds for s also on the assumption that it holds up to $s-1$. But we know that it holds for $s = 1$, and $s = 2$. Hence it holds universally.

§ 4. Rule for fixing the Regular Minors* of A.

Consider the characteristic matrix C of the canonical form of a linear substitution, with the single characteristic-root α, and the invariant-sequence,

$$a_1, a_2, \dots, a_k.$$

It is easily seen that sth regular minor of C is obtained by cancelling in C the

$$1\text{st}, (a_1+1)\text{th}, (a_1 + a_2 + 1)\text{th}, \dots (a_1 + a_2 + \dots + a_{s-1} + 1)\text{th}$$

columns and the

$$a_1\text{th}, (a_1+a_2)\text{th}, \dots, (a_1 + a_2 + \dots + a_s)\text{th}$$

rows.

Also we know that the regular minors of a λ-matrix are relatively invariant with respect to elementary transformations. If C be the canonical form of A, it is clear from the transfarmations of § 2, 3 that the

$$1\text{st}, (s+2)\text{th}, (2s+3)\text{th}, \dots, (\overline{t-1}s+t)\text{th}; (ts+t)\text{th}$$

$$(\overline{t+1}s+t)\text{th}, \dots, (\overline{r-1}s+t)\text{th}$$

columns of C correspond to the

$$1\text{st}, 2\text{nd}, 3\text{rd}, \dots, t\text{th}; (t+1)\text{th}, (t+2)\text{th}, \dots, r\text{th}$$

columns of A respectively and that the

$$(s+1)\text{th}, (2s+2)\text{th}, \dots, (ts+t)\text{th}; (t\overline{s+1}+s)\text{th},$$

$$(t\overline{s+1}+2s)\text{th}, \dots, (rs+t)\text{th}$$

---

* The kth regular minor of a λ-matrix is that kth minor, which gives the greatest common divisor of all the kth minors of the matrix.

rows of C correspond respectively to the

$$t\text{th}, (t-1)\text{th}, (t-2)\text{th}, \ldots 1\text{st} : r\text{th}, (r-1)\text{th}, \ldots (t+1)\text{th}$$

rows of A from the *bottom*. Hence we immediately deduce that

1. The kth regular minor of A $(k \leqslant t)$ is obtained by cancelling in A the first k columns, and the tth, $(t-1)$th, ..., $(t-k+1)$th rows from the *bottom*.

2 The kth regular minor of A $(k > t)$ is obtained by cancelling in A the first k columns and the

$$t\text{th}, (t-1)\text{th}, \ldots, 1\text{st}, r\text{th}, (r-1)\text{th}, \ldots (r-k+t+1)\text{th}$$

rows from the *bottom*.

5. Geometrical Method.

The following geometrical method will serve as a good illustration of the geometrical theory of matrices.* Any n independent reference points $e_1, e_2, \ldots, e_n$ determine a space of the nth order, any point in it being given by $(\alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n)$, where $\alpha_1, \alpha_2, \ldots \alpha_n$ are numbers of the real domain. Let $\triangle$ be the matrix of any collineation in $[n]$ space. The latent roots of the matrix are given by

$$| \triangle - \lambda | = 0.$$

Here we consider only matrices with a single latent root, there being no loss of generality, since a matrix with several distinct latent roots can be expressed as a compartite matrix, each part of which has only one latent root. If x is a point such that $\triangle x = \lambda x$, then x is called a latent point corresponding to the characteristic or latent root λ. If $\Delta y_1 = \lambda y_1 + x$, then y_1 is called a semi-latent point of the first species. If $\triangle y_2 = \lambda y_2 + y_1$ then y_2 is called a semi-latent point of the second species. Similarly the semi-latent points of higher species are defined. It has been shown in Whitehead's *Universal Algebra*

---

* For a fuller exposition of the geometrical theory of matrices, see Ch. VI, Book IV, Vol. I, Whitehead's '*Universal Algebra*.'

that there exists a set of r_1 independent points, which are latent, a second set of r_2 independent points, which are semi-latent of the 1st species; ......; a kth set of r_k independent points, which aresemi-latent of the $(k-1)$th species. It has also been proved that these sets of points are also mutually independent, and that

$$r_1 + r_2 + \dots + r_k = n$$

so that these points can be chosen as reference points. These r_1 independent latent points define a region of order r_1 called 'the latent region'. The r_2 independent semi-latent points of the 1st species, together with the r_1 latent points, define a region of order $(r_1 + r_2)$ called 'the semi-latent region of the 1st species.' Similarly the semi-latent regions of higher species are defined, any region containing all the previous regions, and, finally, the last semi-latent region coinciding with the space $[n]$. The sequence

$$r_1, r_1 + r_2, r_1 + r_2 + r_3, \dots, (r_1 + r_2 + \dots + r_k)$$

has been called the 'sequence of latent orders'*, and the sequence

$$r_1, r_2, \dots r_k$$

the 'latency-sequence.'* Further there is a fundamental relation between the two sequences, the invariant sequence and the latency-sequence, *viz.*, 'The invariant-sequence of a matrix, corresponding to any latent or characteristic root, is the conjugate-sequence of its latency-sequence for the same latent root.'*

It is this important theorem that we intend making use of in this geometrical method.

Let $\triangle$ be the matrix of S. Consider the points

$$e_1\,(1, 0, \dots 0);\ e_2\,(0, 1, 0 \dots 0);\ \dots;\ e_n\,(0, 0, \dots 1).$$

The operation of $\triangle$ on e_μ gives

$$\triangle e_\mu = a_1 e_\mu + a_2 e_{\mu-1} + \dots + a_\mu e_1.$$

---

* *cf.* § 8, 'Invariant Factors and Integer-Sequences' by R. Vaidyanathaswamy. *J. I. M S.*—June, 1924.

Since, in the present case,

$$a_2 = 0 = \dots = a_r, \quad a_{r+1} \neq 0,$$

we have $\Delta e_1 = a_1 e_1 ;\ \Delta e_2 = a_1 e_2 ;\ \dots ;\ \Delta e_r = a_1 e_r.$

Therefore, $e_1 e_2 \dots e_r$ are the independent latent points of S. These r latent points define a region of order r called the 'latent region of S,' corresponding to its single latent root a_1. Again,

$$\Delta e_{r+1} = a_1 e_{r+1} + a_{r+1} e_1 ;\ \Delta e_{r+2} = a_1 e_{r+2} + a_{r+2} e_1 + a_{r+1} e_2 ;$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$\Delta e_{2r} = a_1 e_{2r} + a_{2r} e_1 + \dots + a_{r+1} e_r.$$

Therefore whatever be the values of $a_{r+2}, a_{r+3}, \dots, a_n$,

since $a_{r+1} \neq 0$, $\qquad e_{r+1}, e_{r+2}, \dots, e_{2r}$

are the r independent semi-latent points of the first species. These points, together with the r latent points, define a region of order $2r$, called the 'semi-latent region of the 1st species.' Similarly it can be proved that the semi-latent region of the 2nd species is of order $3r$, the third species of order $4r$, and so on. Hence, if $n = sr + t$, the sequence of latent orders for S is

$$r,\ 2r,\ 3r,\ \dots,\ sr,\ sr + t \qquad (s + 1) \text{ terms} \qquad \dots \quad (1)$$

$\therefore$ the latency-sequence is

$$r,\ r,\ \dots,\ r,\ t \qquad (s + 1) \text{ terms} \qquad \dots \quad (2)$$

Since the invariant-sequence of S is the conjugate-sequence of the sequence (2) by the theorem referred to above, the invariant-sequence of S is

$$s + 1,\ s + 1,\ \dots \text{(up to } t \text{ terms)},\ s,\ s,\ \dots \text{(up to } r - t \text{ terms)}.$$

It thus appears that the invariant-factors of our substitution S depend only on the condition

$$a_2 = 0 = a_3 = \dots = a_r,\ a_{r+1} \neq 0$$

and are entirely independent of the values of

$$a_{r+2}, a_{r+3}, \dots, a_n.$$

AN ANALOGUE OF A THEOREM OF PHRAGMEN-LINDELÖF.

BY S. P. JAIN, M. SC.,
Research Scholar, Allahabad University.

1. The following theorem is only a particular case of the classical Phragmen-Lindelof theorem.*

If

(1.1) *$\phi(z)$ be an integral function of z of order $e^{r\varepsilon(r)}$, where $|z| = r$ and ε is positive and tends to zero as $r \to \infty$,*

(1.2) *$\phi(z) \to 0$ as $|z| \to \infty$ along the real axis,*
then $\phi(z) \equiv 0$.

Dr. P. L. Srivastava proved† an analogue of the above theorem by establishing that the theorem holds when the condition (1.2) is replaced by the condition

$$\int_0^\infty \phi(\pm t)\, dt \text{ converges absolutely.} \qquad \dots \quad (1.3)$$

Evidently, the above theorems fail when $\phi(z) = O(e^{kr})$, k being a positive constant, as may be seen by considering the function $\left(\frac{\sin z}{z}\right)^2$.

It is, however, known that if $\phi(z) = O(e^{kr})$ and if it is bounded on three radii vectores, the angle between no two consecutive vectores being greater than or equal to π, then $\phi(z)$ reduces to a constant. The object of this paper is to obtain an analogue of this theorem similar to Dr. Srivastava's theorem. That is to say, I wish to prove the following theorem.

---

* Sur une extension d'un principe, etc., *Acta Mathematica*, Vol. 31, pp. 386-387.

† 'A theorem on Integral Functions,' *The Allahabad University Studies*, Vol. III, 1931; an abstract of the same appeared in '*Proceedings of the Academy of Science, U.P.*, Vol. I.

If

(1.4) *$\phi(z)$ is an integral function of z of order e_{kr}, where $|z| = r$ and k is any positive constant ;*

(1.5) $\int_0^{\infty(\psi)} \phi(t)\, dt$ *converges absolutely along any three vectores $\psi = \psi_1$, $\psi = \psi_2$, $\psi = \psi_3$ such that the angle between no two consecutive radii vectores taken in the same sense $\geqslant \pi$;*

then $$\phi(z) \equiv 0.$$

I am indebted to Dr. Srivastava for suggesting the problem.

2. The proof of the theorem rests on an application of the Borel's integral.

Let $\phi(z)$ be $= \sum_0^\infty \frac{c_n z^n}{n!}$. By virtue of (1.4), we have

$$c_n = O(k+\varepsilon)^n, \text{ where } \varepsilon > O \text{ is arbitrarily small.} \quad \ldots \quad (2.1)$$

Now, consider the function $f(x)$ defined by

$$f(x) = \int_0^\infty \phi(t)\, e^{-t/x}\, dt \quad \ldots \quad (2,2)$$

$$f(x) = \int_0^\infty \phi(t)\, e^{-t/x}\, dt = \int_0^\infty e^{-t/x} \sum_0^\infty \frac{c_n t^n}{n!}\, dt$$

$$= \sum_0^\infty \frac{c_n}{n!} \int_0^\infty e^{-t/x}\, t^n\, dt = \sum_0^\infty c_n\, x^{n+1},$$

provided $|x| < \frac{1}{k+\varepsilon}$. The inversion of the order of summation and integration involved here is justified by the convergence of

$$\int_0^\infty e^{-t/x} \sum_0^\infty \frac{|c_n t^n|}{n!}\, dt.$$

By (2.1), $f(x)$ is an analytic function of x at least inside

$$|x| = \frac{1}{k+\varepsilon}.$$

Now I wish to show that it is permissible to take the integral along any radius vector issuing from the origin and going to infinity.

Let us consider the vector $\psi = \psi'$, where $|\psi'| < \frac{\pi}{2}$, and let us suppose in the first instance that x is real and positive and less than $\frac{\cos\psi}{k}$.

Since

$$\lim_{\rho\to\infty} \left| \int_0^{\psi'} \phi(\rho e^{i\psi})\, e^{-\frac{\rho e^{i\psi}}{x}} \rho e^{i\psi}\, i d\psi \right| \qquad \left(t = \rho e^{i\psi} \right)$$

$$\leqslant \lim_{\rho\to\infty} \int_0^{\psi'} \rho e^{\rho\left(k - \frac{\cos\psi'}{x}\right)} d\psi$$

and since this $\to 0$ as $\rho \to \infty$, we have

$$\int_0^\infty \phi(t)\, e^{-t/x}\, dt = \int_0^{\infty(\psi')} \phi(t)\, e^{-t/x}\, dt.$$

Now, by the principle of analytic continuation $f(x)$ defined by (2.2) is continued analytically by the integral

$$\int_0^{\infty(\psi')} \phi(t)\, e^{-t/x}\, dt.$$

The integral $\int_0^{\infty(\psi')} \phi(t)\, e^{-t/x}\, dt$ represents an analytic function of x in the region

$$\cos(\psi' - \theta) > kR \qquad \dots \quad (2.3)$$

where $x = R\, e^{i\theta}$.

Similarly, the integral $\int_0^{\infty(\psi'')} \phi(t)\, e^{-t/x}\, dt$ represents an analytic function of x in the region

$$\cos(\psi'' - \theta) > kR. \qquad \dots \quad (2.4)$$

Let ψ'' be taken sufficiently near ψ', so that (2.3) and (2.4) have a common region.

Now, if x is chosen to lie in the region common to (2.3) and (2.4),

the integral $\int_{\psi=\psi'}^{\psi=\psi''} \phi(t)\, e^{-t/x}\, dt$ will vanish as $\rho \to \infty$.

Thus $\int_0^{\infty(\psi'')} \phi(t)\, e^{-t/x}\, dt$ gives the analytic continuation of (2.2).

By continuously varying ψ'', in a similar manner we establish that

$$\int_0^{\infty(\psi)} \phi(t)\, e^{-t/x}\, dt$$

gives the analytic continuation of (2.2) for $-\pi \leqslant \psi \leqslant \pi$.

Now, let us consider $\int_0^{\infty(\psi)} \phi(t)\, e^{-t/x}\, dt$.

Suppose that $|x| \geqslant \delta > 0$; then since

$$\left| \int_0^{\infty(\psi_1)} \phi(t)\, e^{-t/x}\, dt \right| \leqslant \int_0^{\infty} \left| \phi\left(\rho e^{i\psi_1}\right) e^{-\rho \cos(\psi_1 - \theta)/R} \right| d\rho$$

$$\leqslant \int_0^{\infty(\psi_1)} \left| \phi(t) \right| d\rho, \text{ provided } \cos(\psi_1 - \theta) \geqslant 0,$$

and the last integral is convergent by virtue of (1.5), hence $f(x)$ represents an analytic function of x in the region defined by

$$|x| \geqslant \delta > 0, \text{ and } \cos(\psi_1 - \theta) \geqslant 0,$$

and is bounded in this region.

Similarly, we find that $f(x)$ is analytic in the region defined by

$$|x| \geqslant \delta > 0;\ \cos(\psi_2 - \theta) \geqslant 0;$$

and $$|x| \geqslant \delta > 0;\ \cos(\psi_3 - \theta) \geqslant 0;$$

and further $|f(x)|$ is bounded in these regions.

By reason of the choice of ψ_1, ψ_2, ψ_3 these three regions cover the whole plane of z outside a certain triangle, the perpendiculars from the

origin on whose sides may be at most equal to δ. But we already know that $f(x)$ is an analytic function of x inside the circle

$$|x| = \frac{1}{k + \varepsilon}.$$

We can choose δ sufficiently small so that the triangle is included in the above circle. Hence $f(x)$ is an integral function of x and further $f(x)$ is bounded throughout the plane. By Liouville's theorem $f(x)$ reduces to a constant which must be zero for $f(0) = 0$. In other words all the co-efficients of the series $\sum_{0}^{\infty} c_n x^{n+1}$ must vanish, and hence $\phi(z \equiv 0)$.

CYCLIC CURVES ON AN ELLIPSOID.

By S. Mukhopadhyaya.

1. Cyclic Curves and Vertices.

At every point P on a closed convex surface W, possessing the requisite degree of continuity, there exists at least one normal section such that P is a cyclic point (vertex) of the section.

Such a section may be called *cyclic* or to possess *cyclicity* at P. To distinguish it from a section which is a circle, the latter will be called a ***circular*** section.

A ***cyclic curve*** on W will be defined to be a curve such that at every point P of the curve, the tangential normal section of W is cyclic.

At each point P of W there exists, *generally*, a *unique* normal section which is cyclic. But there exist also points Q on W where every normal section is cyclic. Such a point Q will be called a ***vertex*** of W. Professor *W. Blaschke* has suggested, in his lectures on *Selected Problems of Differential Geometry*, delivered at the Calcutta University in February 1932, which are being published by the Calcutta University, that the minimum number of vertices on W may be *six*. It is not the object of this paper to go into this problem. The object of this paper is to study certain properties of cyclic curves on an ellipsoid, which is known to possess *six* vertices.

2. THEOREM: ***Along a cyclic curve on an ellipsoid the central perpendicular on the tangent plane has a constant length.***

Denote by W the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \qquad a>b>c.$$

Let P (x, y, z) be any point of W, *not* at one of the vertices.

Denote by V the central section of W parallel to the tangent plane T at P, and the foot of the perpendicular from P on V by N.

If QPR be the normal section S at P possessing cyclicity at P then evidently the chord QR of V is bisected at N and there exists only one chord of V which is bisected at N (ξ, η, ζ).

The direction cosines of PN are proportional to $\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}$.

Therefore

$$\xi = x + k \cdot \frac{x}{a^2}, \quad \eta = y + k \cdot \frac{y}{b^2}, \quad \zeta = z + k \cdot \frac{z}{c^2} \quad \dots \quad (1)$$

If the direction cosines of t, the tangent to S at P, be l, m, n then

$$\frac{lx}{a^2} + \frac{my}{b^2} + \frac{nz}{c^2} = 0, \quad \frac{l\xi}{a^2} + \frac{m\eta}{b^2} + \frac{n\zeta}{c^2} = 0. \quad \dots \quad (2)$$

From (1) and (2) we have

$$\frac{lx}{a^4} + \frac{my}{b^4} + \frac{nz}{c^4} = 0 \quad \dots \quad (3)$$

If we now replace l, m, n by dx, dy, dz, taken along S, we have

$$\frac{xdx}{a^4} + \frac{ydy}{b^4} + \frac{zdz}{c^4} = 0,$$

whence, by integration

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} = \text{constant}. \quad \dots \quad (4)$$

But if p denote the central perpendicular on T, we have

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} = \frac{1}{p}.$$

Therefore, along a cyclic curve, p is constant.

3. THEOREM : *Along a cyclic curve on an ellipsoid, the Gaussian curvature is constant.*

If ρ and σ denote the radii of principal curvatures at P, then

$$\frac{1}{\rho} = \frac{p}{OQ^2}, \quad \frac{1}{\sigma} = \frac{p}{OR^2},$$

where OQ and OR are the principal semi-diameters of V.

Therefore, $$\frac{1}{\rho\sigma} = \frac{p^2}{OQ^2 \, , OR^2}$$

$$= \frac{p^4}{p^2 . OQ^2 . OR^2} = \frac{p^4}{a^2b^2c^2} = \text{constant.}$$

4. Equations to the Cyclic Curves.

The cyclic curves are the intersections of W with

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} = \frac{1}{p^2}, \quad a > p > c.$$

The cyclic curves therefore lie on the quadric cone

$$\frac{x^2}{a^2}\left(\frac{1}{a^2} - \frac{1}{p^2}\right) + \frac{y^2}{b^2}\left(\frac{1}{b^2} - \frac{1}{p^2}\right) + \frac{z^2}{c^2}\left(\frac{1}{c^2} - \frac{1}{p^2}\right) = 0. \quad \dots \quad (6)$$

For each p there are two cyclic curves, symmetrically situated about the centre of W, each curve being closed.

For, $p = b$, equation (6) becomes

$$\frac{x^2}{a^4}\left(a^2 - b^2\right) - \frac{z^2}{c^4}\left(b^2 - c^2\right) = 0. \quad \dots \quad (7)$$

which reduces to two planes. The corresponding cyclic curves pass through B and B$'$. They are the locus of the points of contact of the two right-circular cylinders circumscribable about W.

The projections of the cyclic curves on $y = 0$ are

$$\frac{x^2}{a^4}\left(a^2 - p^2\right) - \frac{z^2}{c^4}\left(p^2 - c^2\right) = 0. \quad \dots \quad (8)$$

They may be divided into three classes

(i) $a > p > b$, (ii) $b > p > c$, (iii) $p = b$.

Class (i) go round A and A$'$ in gradually dwindling closed ovular circuits of which A and A$'$ are limiting points.

Class (ii), similarly, go round C and C$'$, which are limiting points of these circuits.

Classes (i) and (ii) approach the points B and B′ in a hyperbolic fashion without going round any of them, or passing through them.

Class (iii) consists of two curves each of which passes through B and B′. Each of the classes (i) and (ii) tends to be asymptotic to (iii) in the neighbourhood of the points B and B′. Except the two curves of class (iii) no two curves have a common point.

5. Nature of the Vertices of an Ellipsoid.

I call a cyclic point of a plane curve where the circle of curvature falls below the curve, an *under-cyclic* point and a cyclic point where the circle of curvature falls above the curve an *over-cyclic* point.

A point may, however, happen to exist on a plane curve where an under-cyclic point unites with an over-cyclic point. This will be the case if, at a point, the osculating conic happens to be a circle without being sextactic. Such a point may be called a *circular* point but a more appropriate name would be perhaps *meta-cyclic*.

A point Q of a convex surface W may be called an *under-cyclic vertex*, if every normal section at Q has Q for an under-cyclic point. Similarly a point Q on W may be called an *over-cyclic vertex*, if every normal section at Q has Q for an over-cyclic point. A point Q on W may be called a *meta-cyclic vertex*, if some normal sections at Q have Q for an under-cyclic point and others have Q for an over-cyclic point.

The vertices A and A′ are under-cyclic, C and C′ over-cyclic and B and B′ meta-cyclic.

To avoid circumlocution we shall call the above three classes of vertices as vertices of class (i) class (ii) and class (iii) respectively.

6. Change of Cyclicity of the Cyclic Curves.

Let us find the locus of points on the ellipsoid where normal circular sections exist.

The direction cosines of the two systems of circular sections of the ellipsoid are proportional to

$$c\sqrt{a^2-b^2},\quad 0,\quad \pm\, a\sqrt{b^2-c^2}.$$

The direction cosines of the normal at P (x, y, z) are proportional to

$$\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}.$$

Hence a circular normal section will exist at P if

$$\frac{x}{a^2} \; c\sqrt{a^2 - b^2} \pm \frac{z}{c^2} \, a\sqrt{b^2 - c^2} = 0,$$

that is $$c^3\sqrt{a^2 - b^2}\, x \pm a^3\sqrt{b^2 - c^2}\, z = 0 \qquad \dots \quad (10)$$

which represents two plane curves.

At each point where (10) meets any of the cyclic curves (6), there will exist a normal circular section tangential to the cyclic curve.

If we go round a cyclic curve, we will encounter a change of cyclicity at a point from under to over or over to under, if and only if the normal tangential section at the point is circular. The cyclic curves in the immediate neighbourhood of A and A′ will maintain uniformly under-cyclicity. Those in the immediate neighbourhood of C or C′ will maintain uniformly over-cyclicity.

At B and B′, the curves (10) will divide the immediate neighbourhood into regions where the cyclicity will be different in two contiguous regions but uniform in the same sectional region. The normal sections at B and B′ will therefore vary in cyclicity in turning from one region into another.

The vertices A, A′ thus belong to class (i), C, C′ to class (ii) and B, B′ to class (iii).

To solve Professor Blaschke's Problem we should be able to prove that on a convex oval to each vertex of a given class corresponds another of the same class and that vertices of each of the three classes exist.

REMARKS ON SPURIOUS CORRELATION.

By C. N. Srinivasiengar.

1. The expression obtained by Mr. N. K. Adyanthaya for the correlation (according to the usual definition) between any two functions $f(x_1, x_2)$ and $\phi(x_3, x_4)$ where the variables may be inter-correlated [*J.I.M.S.*, Vol XIX, p. 163, equation (7)] is of considerable theoretical interest. It will be clear that given any function $f(x_1, x_2)$, it will be possible to choose $\phi(x_3, x_4)$ in any number of ways so that their correlation is zero; and in particular, we may have $x_3 = x_1$ and $x_4 = x_2$.

When the correlation between two characters X and Y is zero, the two characters are often spoken of as being *independent*. This nomenclature differs altogether from the sense in which it is used in pure mathematics with reference to two functions. Thus, if x_3, x_4 are given functions of x_1 and x_2, any two functions f and ϕ which are functionally independent, are in general connected by a *correlation*.

2. It is unnecessary to take $f = \phi$, as Mr. Adyanthaya does, to consider spurious correlation. This artifice is used by him to find the standard deviations of f and ϕ. But they are obtainable directly. Mr. Adyanthaya's result will be made more striking when expressed in the following generalised form :

Given three mutually uncorrelated variables x_1, x_2, x_3, there exists in general, an amount of spurious correlation between any two functions $f(x_1, x_2, x_3)$ and $\phi(x_1, x_2, x_3)$.

The value takes the symmetrical form

$$\frac{\sigma_1^2 \frac{\partial f}{\partial \bar{x}_1}\frac{\partial \phi}{\partial \bar{x}_1} + \sigma_2^2 \frac{\partial f}{\partial \bar{x}_2}\frac{\partial \phi}{\partial \bar{x}_2} + \sigma_3^2 \frac{\partial f}{\partial \bar{x}_3}\frac{\partial \phi}{\partial \bar{x}_3}}{\sqrt{\sigma_1^2\left(\frac{\partial f}{\partial \bar{x}_1}\right)^2 + \sigma_2^2\left(\frac{\partial f}{\partial \bar{x}_2}\right)^2 + \sigma_3^2\left(\frac{\partial f}{\partial \bar{x}_3}\right)^2} \times \sqrt{\sigma_1^2\left(\frac{\partial \phi}{\partial \bar{x}_1}\right)^2 + \sigma_2^2\left(\frac{\partial \phi}{\partial \bar{x}_2}\right)^2 + \sigma_3^2\left(\frac{\partial \phi}{\partial \bar{x}_3}\right)^2}}$$

where $\frac{\partial f}{\partial \bar{x}_1}$, for example, means the result of substituting $\bar{x}_1, \bar{x}_2, \bar{x}_3$ for x_1, x_2, x_3 in $\frac{\partial f(x_1, x_2, x_3)}{\partial x_1}$.

For a given function f, the second function ϕ may be chosen in an infinite number of ways so as to make this spurious correlation zero. As an example, if f does not involve x_3, we may take ϕ as $\chi(x_3) + \int \psi(x_2)\, dx_2$ where $\psi(x) = 0$ is any equation admitting $\bar{x}_2$, the mean value of x_2, as a root.

3. The last para of Mr. Adyanthaya's paper admits of easy generalisation:

The correlation between x_1 and x_2 is also equal to that between any two functions $\phi(x_1)$ and $\psi(x_2)$.

ON $\frac{(2n-2)!}{n!(n+r)!}$.

BY P. JAGANNATHAN,
Annamalai University.

1. Let

$$F(n+r) = \frac{(2n-2)!}{n!.(n+r)!}.$$

* Balakram has proved that $F(n)$ is an integer for an infinity of values of n. The object of this note is to show that when r is any fixed positive integer, $F(n+r)$ is also an integer for infinitely many values of n.

LEMMA 1: If a is not a divisor of n, and $a \geqslant r+1$, then

$$\left[\frac{2n-2}{a}\right] \geqslant \left[\frac{n}{a}\right] + \left[\frac{n+r}{a}\right].$$

Let $n = ia + t$, where i and t are positive integers, and $1 \leqslant t \leqslant a-1$.

Then, $$\left[\frac{2n-2}{a}\right] = 2i + \left[\frac{2t-2}{a}\right]$$

and $$\left[\frac{n}{a}\right] + \left[\frac{n+r}{a}\right] = 2i + \left[\frac{t+r}{a}\right].$$

Since $\frac{t+r}{a} < 2$, the required condition will hold if $\frac{2t-2}{a} \geqslant 1$,

or if $$(2t-2)-(t+r) \geqslant -a,$$

which is the same as $$a \geqslant r+2-t.$$

LEMMA 2: If $n = pq$, where p and q are primes greater than r, and $\frac{3}{2}p < q < 2p$,

then $$\sum_{\substack{m=1 \\ p!n}}^{\infty} \left[\frac{2n-2}{p^m}\right] = \sum_{\substack{m=1 \\ p/n}}^{\infty} \left\{ \left[\frac{n}{p^m}\right] + \left[\frac{n+r}{p^m}\right] \right\}.$$

Now

$$\sum_{m=1}^{\infty} \left[\frac{2n-2}{p^m}\right] = \left[\frac{2pq-2}{p}\right] + \left[\frac{2pq-2}{p^2}\right] + \ldots$$

$$= 2q-1+3 = 2q+2,$$

since $$\left[\frac{2q-2/p}{p}\right] \geqslant \left[\frac{3p+1-2/p}{p}\right] = 3.$$

* *J. I. M. S.*, Vol. XVIII, pp. 97—100.

Again, $\sum_{m=1} \left\{ \left[\frac{n}{p^m}\right] + \left[\frac{n+r}{p^m}\right] \right\} = \left[\frac{pq}{p}\right] + \left[\frac{pq+r}{p}\right]$

$$+ \left[\frac{pq}{p^2}\right] + \left[\frac{pq+r}{p^2}\right] + \ldots$$

$$= q + q + 1 + 1 = 2q + 2.$$

$$\sum_{m=1}^{\infty} \left[\frac{2n-2}{q^m}\right] = \left[\frac{2pq-2}{q}\right] + \left[\frac{2pq-2}{q^2}\right] + \ldots$$

$$= 2p - 1 + 1 = 2p,$$

since $\left[\frac{2p-2/q}{q}\right] \geqslant \left[\frac{q+1-2/q}{q}\right] = 1.$

$$\therefore \quad \sum_{m=1}^{\infty} \left\{ \left[\frac{n}{q^m}\right] + \left[\frac{n+r}{q^m}\right] \right\}$$

$$= \left[\frac{pq}{q}\right] + \left[\frac{pq+r}{q}\right] + \left[\frac{pq}{q^2}\right] + \left[\frac{pq+r}{q^2}\right] + \ldots$$

$$= p + p = 2p.$$

Hence the lemma.

LEMMA 3: If $\pi(x)$ denotes the number of primes up to x in the arithmetic progression $ky + l$, where $(k, l) = 1$, and $\phi(k)$ the number of numbers less than and prime to k; then for all $\varepsilon > 0$

$$\pi\,\overline{(1 + \varepsilon\, x)} - \pi(x) \sim \frac{1}{\phi(k)} \; \frac{\varepsilon x}{\log x}.$$

This follows readily from the prime number theorem for arithmetic progressions: (*viz.*)

$$\pi(x) \sim \frac{1}{\phi(k)} \; \frac{x}{\log x}.$$

LEMMA 4: Let

(i) $n = kP^{l+s} - P^l + t$, for every prime $P < r + 1$, where l is the least integer for which $P^l \geqslant r + 1$; t a positive integer less than $P^l - r$ and prime to P, and

$$s = \sum_{m=1}^{l} \left\{ \left[\frac{n}{P^m}\right] + \left[\frac{n+r}{P^m}\right] - \left[\frac{2n-2}{P^m}\right] \right\};$$

(ii) $n = pq$, where p and q are primes greater than r, and $\frac{3}{2}p < q < 2p$.

Then $F(n+r)$ is an integer.

The necessary and sufficient condition that $F(n+r)$ is an integer is that for all primes P

$$\sum_{m=1}^{\infty} \left\{ \left[\frac{2n-2}{P^m}\right] - \left[\frac{n}{P^m}\right] - \left[\frac{n+r}{P^m}\right] \right\} \geqslant 0 \quad \ldots \quad (1)$$

Suppose $s > 0$. Let $1 \leqslant h \leqslant s$. Then

$$\left[\frac{2n-2}{P^{l+h}}\right] = 2kP^{s-h} - 1$$

and $$\left[\frac{n}{P^{l+h}}\right] + \left[\frac{n+r}{P^{l+h}}\right] = kP^{s-h} - 1 + kP^{s-h} - 1,$$

so that $$\left[\frac{2n-2}{P^{l+h}}\right] - \left[\frac{n}{P^{l+h}}\right] - \left[\frac{n+r}{P^{l+h}}\right] = 1$$

$$\sum_{h=1}^{s} \left\{ \left[\frac{2n-2}{P^{l+h}}\right] - \left[\frac{n}{P^{l+h}}\right] - \left[\frac{n+r}{P^{l+h}}\right] \right\} = s. \quad \ldots \quad (2)$$

Hence

$$\sum_{m=1}^{\infty} \left\{ \left[\frac{2n-2}{P^m}\right] - \left[\frac{n}{P^m}\right] - \left[\frac{n+r}{P^m}\right] \right\} = \sum_{m=1}^{l} + \sum_{m=l+1}^{l+s} + \sum_{m=l+s}^{\infty} \geqslant 0.$$

since $$\sum_{m=1}^{l} \left\{ \left[\frac{2 - n}{P^m}\right] - \left[\frac{n}{P^m}\right] - \left[\frac{n+r}{P^m}\right] \right\} = -s$$

by hypothesis (1), $\sum_{m=l+1}^{l+s} = s$ by (2), and $\sum_{m=l+s}^{\infty} \geqslant 0$ by lemma (1).

This is true for all primes P less than $r+1$.

If $s \leqslant 0$, the inequality (1) is evidently true for every $P < r+1$.

By hypothesis (ii) and lemmas (1) and (2), (1) is true for all primes $P \geqslant r + 1$.

Hence $F(n + r)$ is an integer.

2. Now we are in a position to prove our main result.

Let $P_1, P_2, \ldots \ldots P_i$ be the primes in order, less than $r + 1$. Let l_α be the least integer for which $P_\alpha^{l_\alpha} \geqslant r + 1$, and

$$s_\alpha = \sum_{m=1}^{l_\alpha} \left\{ \left[\frac{n}{P_\alpha^m}\right] + \left[\frac{n+r}{P_\alpha^m}\right] - \left[\frac{2n-2}{P_\alpha^m}\right] \right\}.$$

The system of congruences

$$x \equiv -P_\alpha^{l_\alpha} + t_\alpha \pmod{P_\alpha^{l_\alpha + s_\alpha}} \qquad (\alpha = 1, 2, \ldots i) \ \ldots \quad (3)$$

where $$0 < t_\alpha < P_\alpha^{l_\alpha} - r \text{ and } (t_\alpha, P_\alpha) = 1,$$

is always solvable since the modulii are prime to each other.

Let $$Q = P_1^{l_1 + s_1} . P_2^{l_2 + s_2} \ldots \ldots P_i^{l_i + s_i}.$$

If θ is a solution of (3), then $Qm + \theta$ is also a solution.

From Lemma (3) if p is a prime of the form $Qm + 1$ and greater than a certain number g (say), then there is at least one prime q of the form $Qm + \theta$ where $\frac{3}{2} p < q < 2p$.

If $n = pq$, then $x = n$ is a solution of (3). Hence $F(n + r)$ is an integer, since n satisfies all the requirements of lemma (4).

Now there is an infinite number of primes of the form $Qm + 1$, which are greater than g and for every p there is at least one n for which $F(n + r)$ is an integer.

Hence, for every fixed positive integer r, $F(n + r)$ is an integer for infinitely many values of n.

DISTRIBUTION OF HEIGHT, WEIGHT AND CHEST-GIRTH IN MADRAS MALES.

By N. K. Adyanthaya, B. Sc. (Lond.) and
N. Sundararama Sastri, M.A.

1. A good deal of work has, in recent years, been done in various countries in the West in Anthropometry; measurements on human skulls have formed the basis of certain investigations as to the extent of similarity between races. But, unfortunately in India, this line of research has not so far received the attention that it deserves. An attempt is made in this paper to analyse the data of measurements of height, weight and chest-girth in a sample of 1000 males. The data were obtained from the United Life Assurance Company of Madras. The cases were picked out at random from their records, and every care was taken to see that no bias was introduced in the selection of the cases. Out of this sample of 1000 males for each of whom weight in pounds and height and chest-girth in inches were recorded, three correlation tables were constructed—for correlations between height and weight, height and chest-girth and weight and chest-girth respectively. The tables are given at the end. From these tables the distributions of height, weight and chest-girth which form the totals of the arrays and columns were separately considered, and curves were fitted to each of these. The following results were obtained in the case of the distribution of height :—

Distribution of Height.

Height in feet and inches	4′ 8.4″ 4′ 9.4″	4′ 9.4″ 10.4″	10.4″ 11.4″	11.4″ 5′ 0.4″	5′ .04″ 5′ 1,4″	5′ 1.4″ 5′ 2.4″	5′ 2.4″ 5′ 3.4″	5′ 3.4″ 5′ 4.4″	5′ 4.4″ 5′ 5.4″	5′ 5.4″ 5′ 6.4″
Frequency	1	2	6	19	25	76	123	149	167	180

Height in feet and inches	5′ 6·4″ 5′ 7·4″	5′ 7·4″ 5′ 8·4″	5′ 8·4″ 5′ 9·4″	5′ 9·4″ 5′ 10·4″	5′ 10·4″ 5′ 11·4″	5′ 11·4″ 6′ 0·4″	6′ 0·4″ 6′ 1·4″	6′ 1·4″ 6′ 2·4″	6′ 2·4″ 6′ 3·4″	6′ 3·4″ 6′ 4·4″
Frequency	118	77	33	13	3	3	3	1	—	1

Total 1000.

Mean height = 5 ft. 4·95 inches.

Calculating the moments about the mean $\mu_2 = 5{\cdot}339066$ (applying Sheppard's corrections).

$\therefore$ The standard deviation σ of height is 2.31064 inches.

$$\mu_3 = 1.7292404, \qquad \mu_4 = 109.5615735.$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = .0196.$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = 3.8435$$

From these values of β_1 and β_2, it is clear that we have to fit a Pearson curve Type VII to the distribution. For the meanings of the constants of the curve that follow, the reader is referred to Elderton's '*Frequency Curves and Correlation.*'

$$m = \frac{5\beta_2 - 9}{2(\beta_2 - 3)} = 6.05661$$

$$a^2 = \frac{2\mu_2\beta_2}{\beta_2 - 3} = 48.65608$$

$$y_0 = \frac{N\Gamma(m)}{a\sqrt{\pi}\,\Gamma(m - \frac{1}{2})}$$

Referring to the tables of the logarithms of the Γ function,

$$\log y_0 = 2.2705046; \qquad y_0 = 186.46.$$

The equation to the curve will then be

$$y = \frac{186.46}{\left(1 + \frac{x^2}{48.65608}\right)^{6.05661}}.$$

2. In order to find out the goodness of fit, we have to calculate the theoretical frequencies corresponding to the various groups, and compare them with the observed frequencies. This requires the integration of the function

$$y = \frac{186.46}{\left(1 + \frac{x^2}{48.65608}\right)^{6.05661}}$$

over certain definite ranges, which is not an easy matter, since it involves the use of the tables of the incomplete β-function which are not available. So we calculated the ordinates at the mid. points of the various groups for this curve, and used a quadrature formula to get out the frequencies. The quadrature formula used was derived as follows:—

Let y_s be the mid. ordinate correponding to an abscissa x_s and let h be the grouping interval. Then we want to find out

$$\int_{x_s - \frac{1}{2}h}^{x_s + \frac{1}{2}h} y\,dx.$$

Write $x = x_s + \theta h$ where $0 < \theta < 1$; $dx = h d\theta$; and the integral is

equal to
$$h \int_{-\frac{1}{2}}^{\frac{1}{2}} y_\theta d\theta$$

where
$$y_\theta = f(x_s + \theta h)$$

$$= y_s + \theta h \frac{dy_s}{dx} + \frac{\theta^2 h^2}{2!} \frac{d^2 y_s}{dx^2} + \dots \dots$$

Substituting this value for y_θ, the required integral is :

$$h \left[y_s + \frac{h^2}{24} \frac{d^2 y_s}{dx^2} + \dots \dots \right].$$

129090
600 008

But in terms of Everett differences

$$h^2 \frac{d^2 y_s}{dx^2} = \delta^2 y_s - \frac{1}{12} \delta^4 y_s + \frac{1}{90} \delta^6 y_s - \dots \dots$$

Taking into account the first term only,

$$h^2 \frac{d^2 y_s}{dx^2} = \delta^2 y_s = y_{s+1} - 2y_s + y_{s-1}$$

$$\therefore \int_{x_s - \frac{1}{2}h}^{x_s + \frac{1}{2}h} y dx = h \left[y_s + \frac{y_{s+1} - 2y_s + y_{s-1}}{24} \right]$$

$$= \frac{h}{24} (y_{s-1} + 22 y_s + y_{s+1})$$

which is a formula for the frequency in the interval $x_s - \frac{1}{2}h$ and $x_s + \frac{1}{2}h$ in terms of the mid. ordinates.

The expected frequencies as compared with the observed and tested for goodness of fit are as shown below :—

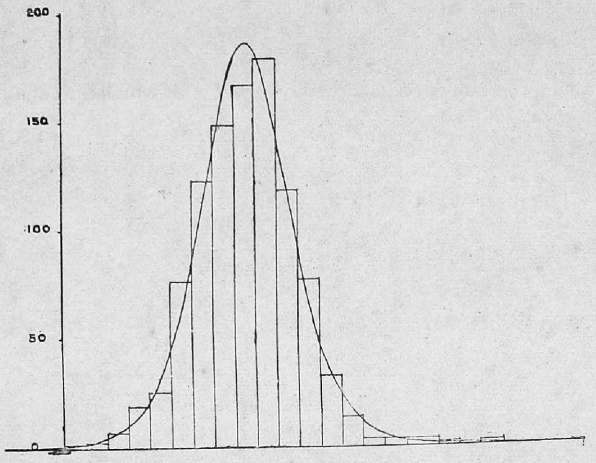
Goodness of Fit of Height Curve.

x	-8.051	-7.051	-6.051	-5.051	-4.051	-3.051	-2.051	-1.051	-0.051	+0.949
y	1.1	2.62	6.23	14.51	32.06	64.58	112.79	162.73	186.31	166.84
Expected frequency	2.2	2.7	6.4	14.9	32.7	65.2	112.9	161.6	184.5	165.6

x	1.949	2.949	3.949	4.949	5.949	6.949	7.949	8.949	9.949	10.949
y	118.22	68.87	34.61	15.7	6.80	2.87	1.21	0.51	0.22	0.10
Expected frequency	118.2	69.5	35.3	16.2	7	3	1.3	0.5	0.2	0.1

$$P = 0.4573.$$

The fit is pretty good. We conclude that in about **47** out of **100** cases we ought to expect a fit worse than the one we have employed.



Distribution of height in a sample of 1000 male adults. Pearson Type VII is fitted to the data. The Histogram shows the observed frequencies.

Scale :— $\frac{1}{5}$ in. = 1 in. of height.

3. We next come to the frequency distribution of weight.

Distribution of Weight.

Weight in Pounds	80.4 87.4	87.4 94.4	94.4 101.4	101.4 108.4	108.4 115.4	115.4 122.4	122.4 129.4	129.4 136.4	136.4 143.4	143.4 150.4
Frequency	3	10	43	64	151	189	178	156	81	55

Weight in Pounds	150.4 157.4	157.4 164.4	164.4 171.4	171.4 178.4	178.4 185.4	185.4 192.4	192.4 199,4	199.4 206.4	206.4 213.4	213.4 220.4
Frequency	23	25	8	6	6	1	—	—	—	1

Total 1000.

Mean weight = 125.46 pounds.

Standard deviation of weight = 16.13 pounds.

$\beta_1 = 0.5915.$

$\beta_2 = 4.6227.$

The appropriate curve is Pearson Type IV for which the equation is

$$y = y_0 \frac{e^{-\nu \tan^{-1} x/a}}{\left(1 + \frac{x^2}{a^2}\right)^m}$$

$$r = \frac{6(\beta_2 - \beta_1 - 1)}{2\beta_2 - 3\beta_1 - 6} = 12.36467.$$

$m = 7.18234.$

$$\nu = \frac{r(r-2)\sqrt{\beta_1}}{\{16(r-1) - \beta_1(r-2)^2\}^{\frac{1}{2}}} = -9.06228.$$

$$a = \frac{\sigma}{4}\sqrt{16(r-1) - \beta_1(r-2)^2} = 6.38379.$$

$\log y_0 = 1.1051133$, and $y_0 = 12.7384$.

The equation to the curve will then be

$$y = 12{\cdot}738 \frac{e^{9.06228 \tan^{-1} x/6.383793}}{\left\{1 + \frac{x^2}{(6.383793)^2}\right\}^{7{\cdot}18234}}$$

The origin for this curve is Mean $+ \frac{\nu a}{r} = 92.7075.$

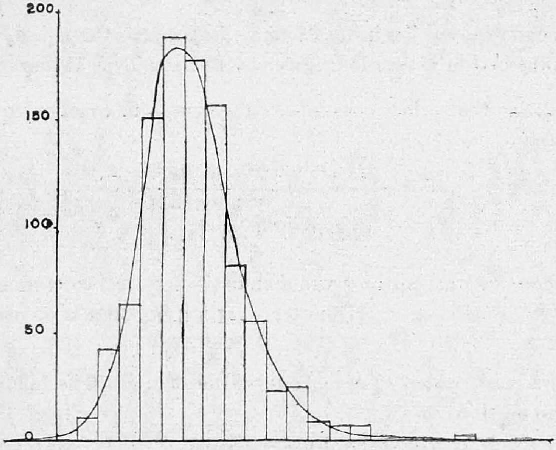
In this case also we calculated the frequencies from the mid. ordinates. A comparison of the expected frequencies with the observed and the goodness of fit are given below :—

Goodness of Fit of Curve of Weight.

x	−1.2582	−0.2582	0.7418	1.7418	2.7418	3.7418	4.7418	5.7418	6.7418	7.7418
y	1.66	8.73	33.00	85.00	149.16	186.43	177.12	137.76	92.58	56.46
Expected frequency	2.5	9.45	34.16	85.51	148.04	184.50	176.19	137.53	92.96	56.95

x	8.7418	9.7418	10.7418	11.7418	12.7418	13.7418	14.7418	15.7418	16.7418	17.7418
y	32.40	17.91	9.71	5.20	2.81	1.53	0.83	0.46	0.26	0.13
Expected frequency	32.80	18.17	9.86	5.29	2.86	1.55	0.89	0.47	0 26	0.15

P=0·102. The fit here is not so good. One of the groups has contributed as much as 5.41 to χ^2. The fit would have been much better but for the large discrepancy in this one group.



Distribution of weight in a sample of 1000 males. Pearson Type IV is fitted to the data. The Histogram shows the observed frequencies.

Scale $\frac{1}{5}$ in. = 7 lbs.

4. Lastly we come to the distribution of Chest-girth.

Distribution of Chest-girth.

Chest-girth in inches	25.4" 26.4"	26.4" 27.4"	27.4" 28.4"	28.4" 29.4"	29.4" 30.4"	30.4" 31.4"	31.4" 32.4"	32.4" 33.4"	33.4" 34.4"	34.4" 35.4"
Frequency	1	1	5	25	77	121	196	202	114	105

Chest-girth in inches	35.4" 36.4"	36.4" 37.4"	37.4" 38.4"	38.4" 39.4"	39.4" 40.4"	40.4" 41.4"	41.4" 42.4"	42.4" 43.4"	43.4" 44.4"	44.4" 45.4"
Frequency	64	38	23	8	11	2	3	2	1	1

Mean chest-girth = 33.09 inches.

Standard deviation of chest-girth = 2.41 inches.

$\beta_1 = .0262, \quad \beta_2 = 4.9243.$

The point (β_1, β_2) lies in the Heterotypic region in the β_1, β_2 diagram, and the best fit is seen to be given by Pearson Type IV curve.

Calculating the various constants of the curve, the equation to the curve is seen to be

$$\frac{185.71 \; e^{.46392 \tan^{-1} x/5.48832}}{\left\{ 1 + \frac{x^2}{(5.48832)^2} \right\}^{4.1019364}} .$$

The fit here is bound to be worse than in the former two cases since the point (β_1, β_2) lies in the Heterotypic region. So, there is no need to examine it.

5. We next proceeded to find out the correlations. The following results were obtained :—

$$r_{hw} = .5144 \pm .0157$$

$$r_{hc} = .3063 \pm .0193$$

$$r_{wc} = .8301 \pm .0066.$$

Here h stands for height, w for weight and c for chest-girth.

The theory of multiple correlation enables one to find out the regression of weight on height and chest-girth or the regression of height or chest-girth on the appropriate associated characters. If a regression equation of weight on height and chest-girth were obtained, that would be the 'best' equation from which to predict weight from a knowledge of height and chest-girth.

If x_0 stand for weight, x_1 for height, and x_2 for chest-girth, we know from the theory of multiple correlation that the required regression equation for weight is

$$\frac{x_0 - \bar{x}_0}{\sigma_0} = -\frac{R_{01}}{R_{00}} \frac{x_1 - \bar{x}_1}{\sigma_1} - \frac{R_{02}}{R_{00}} \frac{x_2 - \bar{x}_2}{\sigma_2},$$

where $\bar{x}_0$ = mean weight, $\bar{x}_1$ = mean height, and $\bar{x}_2$ = mean chest-girth and $\sigma_0, \sigma_1, \sigma_2$ are the standard deviations of weight, height and chest-girth, and R_{00} R_{01}, R_{02} are the minors of the symmetrical determinant

$$R \equiv \begin{vmatrix} 1 & r_{01} & r_{02} \\ r_{10} & 1 & r_{12} \\ r_{20} & r_{21} & 1 \end{vmatrix}$$

The symmetrical determinant to take in this case is

$$\begin{vmatrix} 1 & \cdot 5144 & \cdot 8301 \\ \cdot 5144 & 1 & \cdot 3063 \\ \cdot 8301 & \cdot 3063 & 1 \end{vmatrix}$$

Computing the minors of the determinant and substituting for these and for the means and the standard deviations of the three characters in the regression equation, the regression equation for weight is found to be

$$w = 2.0417\ h + 5.0554\ c - 174.4231.$$

THE JOURNAL

OF

THE INDIAN MATHEMATICAL SOCIETY

NOTES AND QUESTIONS.

ASSOCIATE EDITORS:

M. BHIMASENA RAO; G. A. SRINIVASAN, M.A., L.T.

AND

N. SUNDARAM IYER, M.A.

Volume XIX, No. 11.

October 1932.

Notes and Questions.

A Theorem on Centroids.

In this note the formula given below is established and some of its special cases noticed.

Let P be the mean centre of the points $A_1, A_2, \dots A_n$ for multiples $a_1, a_2 \dots a_n$, and Q the mean centre of the points $B_1, B_2, \dots B_m$ for multiples $b_1, b_2, \dots b_m$; where

$$a_1 + a_2 + \dots a_n = 1 = b_1 + b_2 \dots + b_m \quad \dots \quad (1)$$

then

$$PQ^2 = \Sigma\, a_r\, b_s\, \overline{A_rB_s}^2 - \tfrac{1}{2}\, \Sigma\, a_r\, a_s\, \overline{A_rA_s}^2 - \tfrac{1}{2}\, \Sigma\, b_r\, b_s\, \overline{B_rB_s}^2 \quad \dots \quad (2)$$

where the summation extends to all possible combinations of the values $r, s = 1, 2, 3, \dots$ which have significance.

Proof. Let the position vectors of A_r, B_s relative to any fixed origin be denoted by $\mathbf{A}_r$, and $\mathbf{B}_s$.

Then $$\Sigma\, a_r\, b_s\, \overline{A_r B_s}^2 = \Sigma\, a_r\, b_s\, (\mathbf{A}_r - \mathbf{B}_s)^2$$

$$= \Sigma\, a_r\, \mathbf{A}_r^2 + \Sigma\, b_r\, \mathbf{B}_r^2 - 2\, \Sigma\, a_r\, b_s\, (\mathbf{A}_r \,.\, \mathbf{B}_s)$$

by using (1); and $\dots$ (3)

$$\Sigma\, a_r\, a_s\, \overline{A_r A_s}^2 = \Sigma\, a_r\, a_s\, (\mathbf{A}_r - \mathbf{A}_s)^2$$

$$= 2\, \Sigma\, a_r\, \mathbf{A}^2_r - 2\, \Sigma\, a_r a_s\, (\mathbf{A}_r \,.\, \mathbf{A}_s) \quad \dots \quad (4)$$

Similarly

$$\Sigma\, b_r\, b_s\, \overline{B_r B_s}^2 = 2\, \Sigma\, b_r\, \mathbf{B}_r^2 - 2\, \Sigma\, b_r\, b_s\, (\mathbf{B}_r \,.\, \mathbf{B}_s) \quad \dots \quad (5)$$

Hence, using (1), (3) and (4), we see that the expression on the right in (2) is equal to

$$-2\, \Sigma\, a_r\, b_s\, (\mathbf{A}_r \,.\, \mathbf{B}_s) + \Sigma\, a_r\, a_s\, (\mathbf{A}_r \cdot \mathbf{A}_s) + \Sigma\, b_r\, b_s\, (\mathbf{B}_r \,.\, \mathbf{B}_s)$$

which may be written in the form

$$(a_1 \mathbf{A}_1 + a_2 \mathbf{A}_2 \ldots - b_1 \mathbf{B}_1 - b_2 \mathbf{B}_2 \ldots)^2.$$

Since P and Q have the position vectors $\Sigma a_r \mathbf{A}_r$ and $\Sigma b_r \mathbf{B}_r$ respectively, this proves the result (1).

2. Taking $m = 1$, so that $b_1 = 1$ and $B_1 = Q$, the formula now becomes

$$BP^2 = \Sigma a_r \overline{BA_r}^2 - \tfrac{1}{2} \Sigma a_r a_s \overline{A_r A_s}^2 \qquad \ldots \quad (6)$$

Here B is an arbitrary point, so that choosing it to coincide with P, we have

$$0 = \Sigma a_r \overline{PA_r}^2 - \tfrac{1}{2} \Sigma a_r a_s \overline{A_r A_s}^2$$

that is

$$\Sigma a_r \overline{PA_r}^2 = \tfrac{1}{2} \Sigma a_r a_s \overline{A_r A_s}^2 \qquad \ldots \quad (7)$$

[M'clelland: *Geometry of the Circle*, page 103, Theorem 56.]

Substituting from (7) in (6), we have

$$BP^2 = \Sigma a_r \overline{BA_r}^2 - \Sigma a_r \overline{PA_r}^2$$

[M'clelland, § 55, page 99.]

Lastly let B_1, B_2 ... coincide with A_1, A_2 ...

The formula (2) now gives

$$PQ^2 = -\tfrac{1}{2} \Sigma (a_r - a_s)(b_r - b_s) \overline{A_r A_s}^2 \qquad \ldots \quad (8)$$

If $(X_1\ Y_1\ Z_1)$ are the actual areal co-ordinates of P with respect to the triangle ABC, then it is known that P is the mean centre of A, B, C for multiples $(X_1\ Y_1\ Z_1)$. Hence, from (8) we deduce

$$PQ^2 = -(Y_1 - Y_2)(Z_1 - Z_2)\, a^2 - (Z_1 - Z_2)(X_1 - X_2)\, b^2$$
$$- (X_1 - X_2)(Y_1 - Y_2)\, c^2$$

[Asquith: *Analytical Geometry*, § 261.]

Karnatak College,
Dharwar.

B. B. BAGI.

Method of calculating the 'final yield,' annual or half-yearly, of bonds redeemable at par after a certain number of years.

(1) For the use of businessmen dealing in Stocks and Bonds there has been published in America a book called " Yields of Bonds and Stocks" compiled by David Johnson and others. There are several other books; but they give practically the same results and the book mentioned above is the one that is more generally used.

The figures in the book are worked out on the basis of half-yearly yields, and double of these are taken as the annual yields which are given in the book. Exactness requires that the annual yield instead of being double the half-yearly yield should be equal to the compound interest for two half-years.

(2) This is, however, of minor importance; that which concerns us most is the *method* of calculating the ultimate yield per unit of period, (half-year or year as the case may be), taking into account the gain or loss at redemption.

(3) In the following discussion it is assumed, for facility of expression, that the interest is payable half-yearly and that the unit of value is the Rupee.

Let n = Number of half-years after which the bond is to be redeemed at 100;

c = Cost price per 100 face-value;

p = Interest *paid* every half-year on 100 face-value;

$i = p/c$ = Interest received every half-year on Rupee one invested;

$g = \frac{100 - c}{c}$ = Redemption gain per Rupee *after 'n' half-years*, in the case of bonds purchased at a discount;

y = Redemption gain per half-year per Rupee;

$l = \frac{c - 100}{c}$ = Redemption loss per Rupee *after 'n' half-years*, in the case of bonds purchased at a premium;

z = Redemption loss per half-year per Rupee;

t = Ultimate yield per Rupee per half-year $= i + y$ or $i - z$;

$T = 1 + t$.

(4) For the book figures the formula employed appears to be :

I. $c \times T^n = \frac{p(T^n - 1)}{T-1} + 100$, which gives the same result as formulae II and III below.

II. For Bonds purchased at a discount :—

$$y \times \frac{(T^n - 1)}{(T-1)} = g.$$

III. For Bonds purchased at a premium :—

$$z \times \frac{(T^n - 1)}{(T-1)} = l.$$

There being no direct method of solving equations involving indices higher than 3 the compilers of the book say that they have adopted the process of "trial and error" to arrive at the solution.

(5) I find that the book result can be arrived at in about three (or four) trials from formula II or III by taking

$$T = 1 + i + \frac{g}{n}, \text{ or } = 1 + i - \frac{l}{n},$$

in the first trial, and finding the value of y or z as the case may be.

In the second trial is to be substituted for $\frac{g}{n}$ the value of y, and for $\frac{l}{n}$ the value of z so arrived at, and the new value of y or z is again to be ascertained. This value is again taken for the third trial, and so on.

(6) But the question at issue is not the method of solving the above equations, but whether the *formula itself* is correct. I believe it is not, for the following reason.

The *formula* enables us to find at what rate of interest the *capital amounts to a sum* which is made up of the capital and the accumulation of the interest periodically received if it can be re-invested at a rate combining the rate of paid interest with the redemption gain or the redemption loss. While the *problem* before us is to find the *ultimate periodic yield.* This yield consists of two parts :—

(*a*) the periodic payment of interest, which is a definite sum ; and

(*b*) the part (to be ascertained) which represents the periodic capital increase on account of redemption gain, or the periodic capital decrease on account of redemption loss.

Increase or decrease of capital is not, strictly speaking, increase or decrease of yield, and requires ***separate treatment.***

Increase or decrease of capital depends on the *difference* between the cost price and the redemption value, and on the *time* after which redemption will take place ; it is *quite independent of the rate of interest*, Whereas by the book method the result varies with the rate of interest, and what is more anomalous is that the lower the rate of payable interest the higher the extra yield or redemption gain, so that when there is no interest payable periodically the capital increase is the highest. This will be seen from the following examples :—

Security		Redeemable in years	Half-yearly yield % of interest actually paid	Half-yearly ultimate yield % according to book	Half-yearly extra yield % = Difference between last two columns.
A	0 % @ 80	5	...	2.2565	2.2565
B	3½ % @ 80	5	2.1875	4.2460	2.0585
C	4 % @ 80	5	2.5000	4.5315	2.0315
D	5 % @ 80	5	3.1250	5.1030	1.9780

(7) The fallacy in the book method is the assumption that periodic yield accumulates at a certain rate of interest. This is not the case. Governments or Corporations issuing a loan simply undertake payment of interest at fixed periods, and not payment of interest on interest. If the investor re-invests his periodic receipts he does so at varying rates of interest from time to time. There is no certainty

about the sum to which the investment would amount at compound interest. The only certainty is about the sum periodically paid by way of interest and the sum which will be paid off as principal at the time of redemption—generally Rs. 100—whatever the cost price to the investor may have been; and this, as mentioned above, is not dependent on the interest rate.

(8) I therefore propose a different method.

In the case of securities purchased at a discount the method for calculating the extra yield, (or rather the periodic increase of capital) should be the same as that employed for Post-Office Cash Certificates. Thus in the cases A, B, C, and D above, the cost being 80, the redemption value 100, and the period 10 half-years, the capital increase per half-year is alike in all cases.

If x be the capital increase per one Rupee per one half-year, $(1+x)^{10} = \frac{100}{80}$, $x = \sqrt[10]{\frac{100}{80}} - 1 = .022565$ per Rupee, or 2.2565%.

The 'ultimate yield' (so called) would thus be 2.2565, 4.4440, 4.7565 and 5.3815 % respectively.

In formula form, the capital increase per Rupee per unit of period is

$$\sqrt[n]{\frac{100}{c}} - 1.$$

In the case of securities purchased at a premium, the formula for the decrease of capital per Rupee per unit of period would be :—

$$1 - \sqrt[n]{\frac{100}{c}}.$$

Take for instance, 8 % bond purchased @ 120, and redeemable at par in 10 years (20 half-years), interest payable half-yearly.

The 'current yield' (actually received every half-year) per Rupee is $\frac{4}{120} =$ .033333.

The half-yearly decrease of capital per Rupee by the formula just mentioned would be $1 - \sqrt[20]{\frac{100}{120}} =$ .009093

Net yield per Rupee ... .024240

i.e., 2.4240 %.

The book figure is 2.6935 %.

The above two formulae can also be derived from formula I above, *viz.*:—

$$c \times T^n = \frac{p(T^n - 1)}{T - 1} + 100.$$

By taking $p = 0$, we get $T^n = \frac{100}{c}$ and $T = \sqrt[n]{\frac{100}{c}}$.

In the case of gain on redemption, $T = 1 + t$, and $t = \sqrt[n]{\frac{100}{c}} - 1$

and in ,, loss ,, $T = 1 - t$, and $t = 1 - \sqrt[n]{\frac{100}{c}}$

where 't' represents the extra yield or loss per Rupee per half-year.

I would repeat that in dealing with this problem what we have to keep in mind is that the main question being that of the ultimate yield per unit of period, and one portion of this yield being a definite sum paid periodically by way of interest, the only thing we have to do is to find the extra yield (or capital-increase) per unit of period or the loss-deduction (same as capital-decrease) per unit of period, on account of gain or loss at redemption as the case may be, and that this factor is independent of the rate of paid interest. The only items that matter are the time after which the redemption takes place and the difference between cost price and redemption value, and from these data the periodic capital increase or capital decrease has to be deduced. The only sure test of the correctness of the figure arrived at is its satisfying the formula $c(1 \pm x) = 100$.

Bombay House,
Bruce Street, Fort,
Bombay.

M. N. KUKA.

Lecture Notes.

(1) The conic in areal co-ordinates.

1. Let $S = ax^2 = by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$

be the equation of a conic and let

$$X = \tfrac{1}{2}\frac{\partial S}{\partial x};\ Y = \tfrac{1}{2}\frac{\partial S}{\partial y};\ Z = \tfrac{1}{2}\frac{\partial S}{\partial z}$$

$$\xi = Y - Z;\quad \eta = Z - X;\quad \zeta = X - Y;$$

so that

$$\xi + \eta + \zeta = 0. \qquad \ldots \quad (1)$$

Also we write

$$b + c - 2f = p;\ c + a - 2g = q;\ a + b - 2h = r$$

and

$$\lambda = a + f - g - h = \tfrac{1}{2}(q + r - p);$$
$$\mu = b + g - f - h = \tfrac{1}{2}(r + p - q);$$
$$\nu = c + h - f - g = \tfrac{1}{2}(p + q - r).$$

The conic is an ellipse, parabola, or hyperbola according as

$$\mu\nu + \nu\lambda + \lambda\mu > = \text{ or } < 0.$$

In particular if the conic is a parabola $\Sigma \dfrac{1}{\lambda} = 0$ and the point at infinity on the curve is $\left(\dfrac{1}{\lambda}, \dfrac{1}{\mu}, \dfrac{1}{\nu}\right)$.

2. Conjugate diameters.

Since the lines $\xi = 0$, $\eta = 0$, $\zeta = 0$, pass through the centre of S, the equation $A\xi^2 + 2H\xi\eta + B\eta^2 = 0$ represents a pair of diameters. Using the relations $\zeta^2 = -\zeta\eta - \zeta\xi$, $\eta^2 = -\eta\xi - \eta\zeta$ which are consequences of (1), this may be thrown into the standard form

$$k_1\eta\zeta + k_2\zeta\xi + k_3\xi\eta = 0. \qquad \ldots \quad (2)$$

It is also easy to show that in this form, the condition that the lines $(k_1 k_2 k_3)$ and $(l_1 l_2 l_3)$ should be harmonic is that

$$\sum_{1,2,3} k_1 (l_2 + l_3 - l_1) = 0 = \sum_{1,2,3} l_1 (k_2 + k_3 - k_1) \qquad \ldots \quad (3)$$

It is worth noting that (2) is the equation of the polars with respect to S of the points at infinity on $k_1 yz + k_2 zx + k_3 xy = 0$. In particular the polars with respect to S of the circular points are given by

$$a^2\eta\zeta + b^2\zeta\xi + c^2\xi\eta = 0; \qquad \ldots \quad (4)$$

The pair of diameters (2) will be conjugate, if they are harmonically related to the asymptotes whose equation can be thrown into the form

$$p\eta\zeta + q\zeta\xi + r\xi\eta = 0. \quad \dots \quad (5)$$

Hence (2) represents a pair of conjugate diameters of S if $\Sigma\, k_1(q + r - p) = 0$, that is, if

$$\lambda k_1 + \mu k_2 + \nu k_3 = 0. \quad \dots \quad (6)$$

3. The Axes of a central conic.

Assuming that the diameters (2) are conjugate, they are the polars of the points at infinity on the diameters. Also the polars of the circular points at infinity are given by the equation (4).

The pair of diameters (2) are perpendicular, if they are harmonically related to the diameters (4) the condition for which is $\Sigma\, k_1(-a^2+b^2+c^2)=0$, that is,

$$\Sigma\, k_1 \cot A = 0 \quad \dots \quad (7)$$

The equation of the principal axes of S is obtained by eliminating k_1, k_2, k_3 from (2), (6) and (7). The required equation is, therefore,

$$\begin{vmatrix} \eta\zeta & \zeta\xi & \xi\eta \\ \lambda & \mu & \nu \\ \cot A & \cot B & \cot C \end{vmatrix} = 0$$

(*Cf.* Jones: *Algebraic Geometry*, page 505).

The other form

$$\begin{vmatrix} \xi^2 & \eta^2 & \zeta^2 \\ p & q & r \\ \sin^2 A & \sin^2 B & \sin^2 C \end{vmatrix} = 0$$

can be easily deduced from this.

4. Rectangular hyperbola.

The polars of the points at infinity of the asymptotes are given by equation (5). So the asymptotes will be at right angles if (5) and

(4) are harmonic ; *i.e.*, if

$$\Sigma\, p\,(-a^2 + b^2 + c^2) = 0 = \Sigma\, p \cot A,$$

or
$$\Sigma\, a^2\,(-p + q + r) \quad = 0 = \Sigma\, a^2\lambda \qquad \ldots \quad (8)$$

Thus the condition for a rectangular hyperbola can be thrown into either of the above forms.

Axis of a Parabola.

The point at infinity of the chord $lx + my + nz = 0$ is P $[(m - n),\ (n - l),\ (l - m)]$. The polar of this point, with respect to the parabola S, *viz.*,

$$\Sigma\,(m - n)\,X = 0 \qquad \ldots \quad (9)$$

is the equation of the diameter bisecting the above chord. This diameter passes through the point at infinity $Q\left(\frac{1}{\lambda}, \frac{1}{\mu}, \frac{1}{\nu}\right)$ on the parabola Hence the chord $\Sigma\, lx = 0$ and the diameter (9) will be at right angles if P and Q are conjugate points with respect to a circle, *e.g.*,

$$\Sigma\, x^2 \cot A = 0.$$

The condition for perpendicularity is

$$\sum \frac{m - n}{\lambda} \cot A = 0. \qquad \ldots \quad (10)$$

Eliminating $(m - n)$, etc., from (9) and (10) and

$$\Sigma\,(m - n) = 0,$$

we obtain the equation of the axis in the form

$$\begin{vmatrix} X & Y & Z \\ \frac{\cot A}{\lambda} & \frac{\cot B}{\mu} & \frac{\cot C}{\nu} \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

or
$$\Sigma\, \xi\, \frac{\cot A}{\lambda} = 0.$$

(*Cf.* Jones: *Algebraic Geometry*, page 506).

S. D. College,
Lahore,

HUKAM CHAND.

(2) On the Determination of the Real Foci of a Real Conic.

The method of determining the *real* foci of a real conic (*i.e.*, a conic given by an equation whose co-efficients are all real) described below, though not new, is not given in any of the standard text-books on the subject. It is based on the following obvious lemma:

If the straight line

$$x + iy - (x' + iy') = 0 \qquad \ldots \quad (1)$$

touch a real conic then (x', y') is one of its *real* foci.

For, if (1) touches a real conic, then the conjugate imaginary line

$$x - iy - (x' - iy') = 0 \qquad \ldots \quad (2)$$

must also touch the same conic. Hence the equation to the conic must be of the form

$$[x + iy - (x' + iy')]\,[x - iy - (x' - iy')] = (lx + my + n)^2$$

i.e.,
$$(x - x')^2 + (y - y')^2 = (lx + my + n)^2 \qquad \ldots \quad (3)$$

It is clear (x', y') is a focus of (3).

To determine, therefore, the real foci of the real conic

$$S \equiv ax^2 + 2\,hxy + by^2 + 2gx + 2\,fy + c = 0 \qquad \ldots \quad (4)$$

we have to find the values of $n \equiv x' + iy'$ for which

$$x + iy - n = 0 \qquad \ldots \quad (5)$$

touches (4).

By making suitable substitutions in the tangential equation of (4), the condition that (5) should touch (4) may be written

$$A - B + C\,n^2 - 2\,F\,in - 2\,G\,n + 2\,H\,i = 0$$

or
$$C\,n^2 - 2\,n\,(G + i\,F) + (A - B + 2\,i\,H) = 0. \qquad \ldots \quad (6)$$

When $C \neq 0$, *i.e.*, when $S = 0$, is a central conic, equation (6) has two roots given by

$$C\,n = (G + iF) \pm \surd\{\,\Delta\,(a - b) + 2\,ih\,\Delta\,\} \qquad \ldots \quad (7)$$

If $$(\lambda + i\mu)^2 = \triangle(a-b) + 2\,ih\,\triangle$$

then $$\lambda^2 - \mu^2 = \triangle(a-b),\ \lambda\mu = \triangle h \qquad \ldots \quad (8)$$

whence $$\lambda^4 - \triangle(a-b)\,\lambda^2 - \Delta^2 h^2 = 0 \qquad \ldots \quad (9)$$

This quadratic in λ^2 has plainly two real roots having opposite signs. We also notice that their product is $-\triangle^2 h^2$. It follows that equation (9) regarded as a biquadratic in λ has one positive root α (say), one negative root $-\alpha$, and two imaginary roots $i\beta$ and $-i\beta$ where

$$\alpha\beta = h\Delta. \qquad \ldots \quad (10)$$

Thus (7) may be written in either of the ways

$$Cn = G + iF + (\alpha + i\beta)$$

and $$Cn = G + iF - (\alpha + i\beta)$$

Corresponding to these, we have the two real foci of $S=0$, namely,

$$\left\{\frac{G+\alpha}{C}, \frac{F+\beta}{C}\right\}, \left\{\frac{G-\alpha}{C}, \frac{F-\beta}{C}\right\}.$$

Having determined the real foci, the imaginary ones may be obtained in the following manner:

When the two values of n are set in (5), we have the two tangents to $S = 0$

$$x + iy - (G + iF + \alpha + i\beta)/C = 0; \qquad \ldots \quad (11)$$

and $$x + iy - (G + iF - \alpha - i\beta)/C = 0; \qquad \ldots \quad (12)$$

The imaginary conjugate lines

$$x - iy - (G - iF + \alpha - i\beta)/C = 0 \qquad \ldots \quad (13)$$

and $$x - iy - (G - iF - \alpha + i\beta)/C = 0 \qquad \ldots \quad (14)$$

must also touch $S = 0$.

Now (11) and (14) may be written thus:

$$[x - (G + i\beta)/C] + i\,[y - (F - i\alpha)/C] = 0 \qquad \ldots \quad (15)$$

and $$[x - (G + i\beta)/C] - i\,[y - (F - i\alpha)/C] = 0 \qquad \ldots \quad (16)$$

Hence an imaginary focus of S = 0 is

$$\left\{ \frac{G + i\beta}{C}, \frac{F - i\alpha}{C} \right\}.$$

The remaining focus must be

$$\left\{ \frac{G - i\beta}{B}, \frac{F + i\alpha}{C} \right\}.$$

When C = 0, S = 0 represent a parabola. Equation (6) then reduces to

$$2(x' + iy') = (A - B + 2iH)/(G + iF)$$

$$= \frac{AG - BG + 2HF}{F^2 + G^2} + \frac{i(-AF + BF + 2GH)}{F^2 + G^2}.$$

Now since C = 0, $G^2 = -b\Delta$, $F^2 = -a\Delta$.

Hence, when these substitutions are made, we get

$$(a + b)x' = -g + \tfrac{1}{2}[b(A + B)]/G,$$
$$(a + b)y' = -f + \tfrac{1}{2}[a(A + B)]/F.$$

The method can be easily extended to oblique axes. It does not, however, lead to neat results when applied to the general equation of the second degree. We may, however, illustrate it by finding the focus of the parabola $y^2 = 4ax$.

It is easy to prove that if

$$x + my - (x' + my') = 0 \qquad \ldots \quad (17)$$

where $m = \cos\omega + i\sin\omega$, touches a real conic, then (x', y') is a real focus. Now (17) touches the parabola, if

$$x' + my' = -am^2$$

i.e., if $x' + (\cos\omega + i\sin\omega)y' = -a(\cos 2\omega + i\sin 2\omega)$

$\therefore$ $x' + y'\cos\omega = -a\cos 2\omega$; $y'\sin\omega = -a\sin 2\omega$.

$\therefore$ $x' = a,\ y' = -2a\cos\omega$.

Karnatak College,
Dharwar, B. B. BAGI.

Solutions.

Question 1523.

(A. NARASINGA RAO):—Show that the circum-centre of any triangle circumscribed to a tricusp hypocycloid is the ortho-centre of the triangle formed by the mid. points of the tangent-chords.

Solution by B. Ramamurti.

Let us prove first that the tangent at any point (θ_1) meets the curve again in the two points

$$\pi - \frac{\theta_1}{2}, \; 2\pi - \frac{\theta_1}{2}.$$

The points of intersection of the tangent at θ_1 with the curve are given by

$$(2 \cos \theta + \cos 2\theta) \sin \frac{\theta_1}{2} + (2 \sin \theta - \sin 2\theta) \cos \frac{\theta_1}{2} = \sin \frac{3\theta_1}{2}.$$

This reduces to.

$$2 \cos \frac{3\theta}{2} \sin \frac{\theta_1 - \theta}{2} = 2 \cos \left(\theta_1 + \frac{\theta}{2}\right) \sin \frac{\theta_1 - \theta}{2}$$

which gives

$$\theta = \theta_1, \;\; \pi - \frac{\theta_1}{2}, \;\; 2\pi - \frac{\theta_1}{2}.$$

Hence the points of intersection P'_1 $(x'_1 \, y'_1)$, P''_1 $(x''_1 \, y_1'')$ are

$$x_1' = b\left(-2 \cos \frac{\theta_1}{2} + \cos \theta_1\right), \quad y_1' = b\left(2 \sin \frac{\theta_1}{2} + \sin \theta_1\right);$$

$$x_1'' = b\left(2 \cos \frac{\theta_1}{2} + \cos \theta_1\right), \quad y_1'' = b\left(-2 \sin \frac{\theta_1}{2} + \sin \theta_1\right).$$

Hence if the co-ordinates of the mid. point of $P_1' \, P_1''$ be $(\bar{x} \; \bar{y})$

$$\bar{x} = b \cos \theta_1 \quad \bar{y} = b \sin \theta_1.$$

Hence $\frac{\bar{y}}{\bar{x}} = \tan \theta_1$ that is, the mid. point is the point P'.

LIST OF JOURNALS & BOOKS RECEIVED IN THE LIBRARY

during the months of August and September 1931.

1 Abhandlungen aus dem Mathematichen Seminar Hamburg, **8**, 4.
2 American Mathematical Monthly, **38**, 6.
3 Annales de l'ecole Normale Superieure, July and August 1931.
4 Astrophysical Journal, **73**, 5, **74**, 1 .
5 Bulletin of the American Mathematical Society, **37**, 7.
6 Bulletin of the Calcutta Mathematical Society, **23**, 2.
7 Bulletin des Sciences Mathematiques, **55**, June 1931.
8 Crelle's Journal, **164**, 4, **165**.
9 Jahresbericht der Deutsce Mathematiker Vereinigung, **40**, 9, 10 11, 12.
10 Mathematische Annalen, **105**, 1, 2.
11 Monthly Notices of the Royal Astronomical Society, **91**, 7, 8.
12 Philosophical Magazine, **12**, 75, 76, 77, 78.
13 Philosophical Transactions of the Royal Society, **230**, pp. 149 to 220.
14 Proceedings of the Edinburgh Mathematical Society, **2**, 4.
15 Proceedings of the Royal Society of London, **132**, 819, 820, **133**, 821.
16 Proceedings of the London Mathematical Society, **32**, 4, 5, 6 (3 copies each).
17 Proceedings of the Cambridge Philosophical Society, **27**, 3.
18 Rendiconti del Circolo Matematico de Palermo, **55**, 2.
19 Tohoku Mathematical Journal, **34**, 1.
20 Transactions of the Royal Society of South Africa, **19**, 4,

Books.

1 Commemoration Volume of the Calcutta Mathematical Society (Present) Vol. No. 20.
2 Encyklopädie der Mathematischen Wissenschaften, Band III_2 Heft **11**.
3 Expansion of Function in Series of Functions generalizing the Gamma Function. By John Smylie Morrel.
4 Equimodal Frequency Distributions. By Edwin Dubose Monzon.
5 Applications of an Abstract Existence Theorem to both Differential Difference Equations. By Oliver Kenneth Bower.

Numbers in black type refer to the volumes, and those in ordinary type to the numbers of the issues.

The Indian Mathematical Society

Founded in 1907 for the Advancement of Mathematical Study and Research in India.

THE COMMITTEE.

President:

M. T. NARANIENGAR, M.A., Malleswaram, Bangalore.

Treasurer:

S. R. RANGANATHAN, M.A., Librarian, Madras University Library, Chepauk, Madras.

Librarian:

V. B. NAIK, M.A., Professor of Mathematics, Fergusson College, Poona.

Secretaries:

R. VAIDYANATHASWAMY, M.A., D.Sc., Reader, University of Madras, Madras.

MUKAND LAL, M.A., Professor of Mathematics, Government College, Lahore.

Members:

A. C. BANERJI, M.A., M.Sc., F.R.A.S., Professor of Mathematics, Allahabad University, Allahabad.

S. B. BELEKAR, M.A., Professor of Mathematics, College of Science, Nagpur.

T. V. MONE, M.A., Professor of Mathematics, Robertson College, Jubbulpore.

K. R. RAMASWAMY IYENGAR, M.A., L.T., Prof. of Mathematics (Retd.), College of Engineering, Guindy, Madras.

N. M. SHAH, M.A., Principal and Prof. of Mathematics, M. T. B. Arts College, Surat.

PANDIT HEMRAJ, M.A., Principal, Dyal Singh College, Lahore.

G. S. MAHAJANI, M.A. (Cantab.), Ph.D., Principal and Prof. of Mathematics, Fergusson College, Poona.

OTHER OFFICERS.

Assistant Secretary:

S. MAHADEVA AIYAR, M.A., L.T., Assistant Professor of Mathematics, Presidency College, Madras.

Joint Librarian:

S. B. BONDALE, M.A., Professor of Mathematics, Fergusson College, Poona.

Head-Quarters: Poona.